

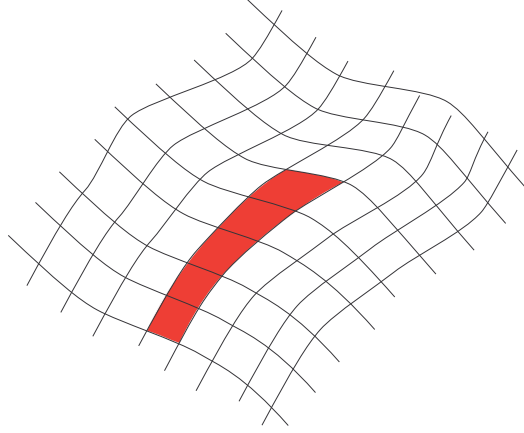
On geometry behind Birkhoff theorem

Pavol Ševera

Dept. of Theoretical Physics, Comenius University,
84215 Bratislava, Slovakia

1 Area as the affine parameter

Suppose N is a 2dim spacetime, i.e. a surface with a metric tensor of signature $(1, 1)$. It is fairly easy to find the isotropic geodesics of N – we just integrate the isotropic directions. It is perhaps more interesting to find affine parameters for these geodesics: one can use the area of a strip between two infinitesimally close integral curves:



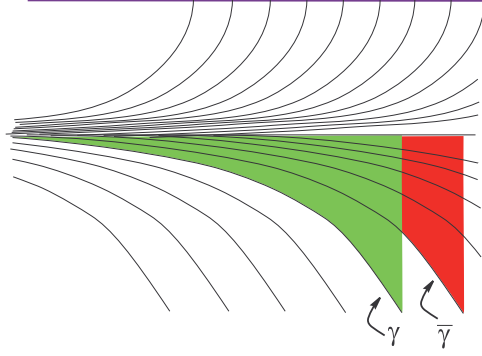
To prove this fact we use the following useful characterization of geodesics in a pseudo-Riemannian manifold M (for the purpose of teaching of general relativity, it appears as a convenient definition): *If we choose coordinates near a curve γ so that the metric tensor is a constant plus $O(r^2)$, where r is (say) the Euclidean distance to γ , then γ is a geodesic iff it is a straight line.* Now our claim that area can be used as the affine parameter is clear, since it is surely so for constant metric tensor.

We shall use our result in the next section to prove Birkhoff theorem, but now, as a digression, we mention some other elementary applications. We can use it conveniently to check isotropic geodesic completeness of 2dim spacetimes. Consider, for example, the Eddington-Finkelstein metrics:

$$ds^2 = -(1 - \frac{2m}{r})du^2 + 2dudr.$$

We prove that the geodesic γ on the following picture is incomplete to the left; we do not use the exact form of the metrics, only the fact that it is invariant

w.r.t. horizontal translations and that the geodesics converge on the left (on the picture, u is the horizontal coordinate):

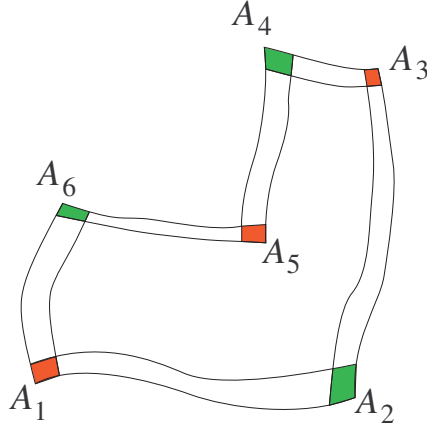


Indeed, the green area between γ and $\bar{\gamma}$ is equal to the red area: to see it, just take the green triangle and translate it to the right. The red area is finite, q.e.d.

As another example, consider a $2n$ -gon with isotropic sides. Widen each of its sides to an infinitesimally narrow strip between two isotropic curves, and compute the expression

$$\mu = \frac{A_1 A_3 \dots A_{2n-1}}{A_2 A_4 \dots A_{2n}},$$

where A_i is the area at the i 'th corner, as on the picture:



μ is clearly independent of the choice of the widening of the sides: if we widen the i 'th side in a different way, A_i and A_{i+1} get multiplied by the same number, so that μ doesn't change. We leave it as an exercise to the reader to prove that μ is actually the result of the parallel transport along the polygon, so that it is equal to the exponential of \pm the integral of the curvature inside the polygon.

2 Birkhoff theorem

Suppose M is a 4dim spacetime on which $SO(3)$ acts by isometries, so that all the orbits are spheres. Birkhoff theorem states that under some assumption on the Ricci tensor, there is a 1-parameter group of isometries of M , commuting with $SO(3)$.

As one easily sees, M is of the form $M = N \times S^2$, where N is a surface, and

$$ds_M^2 = ds_N^2 + r^2 ds_{S^2}^2,$$

where $r > 0$ is a function on N and ds_{M,N,S^2}^2 are the metrics on M , N and the unit sphere. Let R and Ric be the Riemann and Ricci tensor on M , and ω_N the area form on N . Finally, let X_r be the Hamiltonian vector field on N generated by r with symplectic form ω_N , i.e.

$$dr = \omega_N(\cdot, X_r).$$

Birkhoff theorem: If $Ric(v, v) = 0$ for any isotropic v tangent to N , then X_r is a Killing vector field on M .

The proof is split into two lemmas:

Lemma 1: If N is a 2dim spacetime then a vector field w on N is conformal iff for any isotropic geodesic γ , $\omega_N(\dot{\gamma}, w)$ is constant along γ .

Lemma 2: Under the same assumptions as in Birkhoff theorem, if γ is any isotropic geodesic in N and s is its affine parameter, then $dr/ds = \text{const.}$ along γ .

Proof of the theorem: We have to prove that the flow of X_r preserves r and ds_N^2 . First we show that X_r is conformal on N . It follows from the lemmas, since

$$\text{const.} = dr/ds = \langle \dot{\gamma}, dr \rangle = \omega_N(\dot{\gamma}, X_r).$$

It remains to check that X_r preserves ω_N and r , but it certainly does, since it is a Hamiltonian vector field q.e.d.

Proof of Lemma 1: This follows immediately from our result in §1. Indeed, w is conformal iff its flow transports isotropic curves to isotropic curves. As we noticed, the area between γ and an infinitesimally close curve $\bar{\gamma}$ is a linear function of s iff $\bar{\gamma}$ is isotropic q.e.d.

Proof of Lemma 2: If γ is an isotropic geodesic in N and $P \in S^2$ then $\gamma \times \{P\}$ is a geodesic in M . It follows from the $O(2)$ -symmetry of isometries of S^2 preserving P : a geodesic is uniquely determined by its velocity at a point, so that if the velocity is $O(2)$ invariant, the geodesic must be pointwise $O(2)$ -invariant, i.e. it lies in $N \times \{P\}$.

If Q is a different point in S^2 then $\gamma \times \{Q\}$ is also a geodesic; therefore, if we take Q infinitesimally close to P we get that any constant vector $a \in T_P S^2$ satisfies the Jacobi (geodesic deviation) equation

$$\ddot{a} + R(a, \dot{\gamma})\dot{\gamma} = 0.$$

On the other hand, the parallel transport along $\gamma \times \{P\}$ must be $O(2)$ -equivariant, i.e. it acts as a multiple of the identity on $T_P S^2$. It also preserves lengths, so that a/r is parallel, and

$$\ddot{a} = (ra/r)'' = \ddot{r}a/r.$$

To prove that $\ddot{r} = 0$ it remains to show that if v is an isotropic vector tangent to N and a is a vector tangent to S^2 , then $R(a, v)v = 0$. By definition, $Ric(v, v)$ is the trace of the linear map A defined by $A(w) = R(w, v)v$. From $O(2)$ symmetry we see that $A(a) = \lambda a$ for some number λ (independent of a); on the other hand, A restricted to TN is nilpotent, since $A(v) = 0$ and $A(w)$ is orthogonal to v for any w , hence it is a multiple of v . Therefore $0 = Tr A = 2\lambda$, i.e $A(a) = 0$, q.e.d.